Basic Modal Logic

1. Note the following intuitive equivalency among the two standard *alethic* modalities (the modes of possibility and necessity):

$$\sim \text{Nec } \varphi \approx \text{Poss } \sim\varphi$$
And
$$\sim \text{Poss } \varphi \approx \text{Nec } \sim\varphi$$

And note the same basic equivalence among the two standard *deontic* modalities (the modes of obligation and permission):

$$\sim \text{Obl } \varphi \approx \text{Perm } \sim\varphi$$
And
$$\sim \text{Perm } \varphi \approx \text{Obl } \sim\varphi$$

In turn, these equivalencies resemble the familiar quantificational equivalencies:

$$\sim\forall x \ \varphi(x) \approx \exists x \sim\varphi(x)$$
And
$$\sim\exists x \ \varphi(x) \approx \forall x \sim\varphi(x)$$

2. Modal logic is meant to capture seeming entailments between such alethic and deontic notions. In basic modal logic we have two new sentential operators. The <u>strong modal operator</u> is symbolized by the box (□), while the <u>weak modal operator</u> is symbolized by the diamond (◊).

3. The syntax for these operators is quite simple: if φ is a wff, then both □φ and ◊φ are both wffs.

4. So what about the semantics for modal logic? Following Kripke, a semantics for modal logic can be understood in terms of <u>a framework of world-models</u>.

Here's how it works:

(1) We begin with a constellation of distinct "worlds," one of which is often designated as @. This world corresponds to the "actual" one, or at least the one from which a sequent or argument is evaluated.

(2) These worlds are related to one another by an accessibility relation (which is often diagrammed by arrows). The relation tells us which worlds are "accessible" to which.

(3) In modal semantics, truth is now *intensional*. That is, this semantics doesn't assign truth values directly to atomic sentences, but rather assigns truth to an atomic sentence relative to a world.

(4) Relative to a world, all of the standard sentential and quantificational operations proceed as before. The only thing we need to worry about are the two modal operations:

Box-rule: $|\Box\varphi| = T$ in a world w_i just in case $|\varphi| = T$ in all worlds accessible to w_i .

Diamond-rule: $|\Diamond\varphi| = T$ in a world w_i just in case $|\varphi| = T$ in some world accessible to w_i.

5. Now this is where things get fun. Different kinds of modal logic distinguish themselves with respect to the constraints they place on the accessibility relation. You might have noted that as I described the accessibility relation above, I didn't require that world be accessible to themselves (that is, that the accessibility relation be reflexive). As a result, it is possible for $\Box\varphi$ to be true in some world without φ being true in it. Now this would be weird indeed if the strong modal operator is supposed to correspond to alethic necessity. If some proposition is necessary, then it dang well ought to be true! However, that isn't nearly so weird if the strong modal operator corresponds to the deontic notion of obligation. For the fact that I am obliged to bring about some state of affairs by no means entails that I will actually do so! So at this level of generality, it is best not to place any constraint on the notion of necessity at all.

6. The system of logic that results when no constraints are placed on the accessibility relation is sometimes called K (in honor of Kripke). In the system K, the following schema holds:

The K-schema: $\Box(\varphi\rightarrow\psi) \models (\Box\varphi\rightarrow\Box\psi)$

System K is a very weak system of modal logic, probably too weak to be of any lasting interest to logicians. In addition to K, logicians have constructed stronger systems of modal logic. Here are some of the best known:

System D: The accessibility relation is serial. Hence $\Box\varphi\models\Diamond\varphi$. This system is often taken to characterize the basic deontic logic.
System T: The accessibility relation is reflexive. Hence $\Box\varphi\models\varphi$.
System B: The accessibility relation is both reflexive and symmetric. Hence $\Diamond\Box\varphi\models\varphi$
System S4: The accessibility relation is reflexive and transitive. Hence $\Box\varphi\models\Box\Box\varphi$.
System S5: The accessibility relation is reflexive, symmetric, and transitive. Hence $\Diamond\varphi\models\Box\Diamond\varphi$.

Far and away, S5 is the best known system of modal logic. This is the one in which the accessibility relation essentially sorts worlds into equivalence classes. It's also the one you'd get if each and every world were accessible to each other.

7. By the way, the names S4 and S5 derive from C.I. Lewis, an early 20th-century pioneer in modal logic. Lewis is well-known for trying to define a modally robust conditional that would avoid many of the "paradoxes" (or at least infelicities) of the material conditional. His "Lewis-hook" was defined as follows:

Def-Hook: $\varphi\Rightarrow\psi$ just in case $\Box(\varphi\rightarrow\psi)$

Unfortunately, Lewis's hook is defined in terms of the material conditional. If we wanted a completely sui generis conditional that altogether avoids the material conditional and its attendant infelicities, we'd perhaps prefer a more direct definition. Perhaps I'll tell you later how you might try to do that.

8. There are other systems of modal logic as well. Epistemic logic, for example, includes a propositional operator K, which symbolizes that that proposition is known. Tense logic, brings in propositional operators F and P, corresponding to whether a given proposition has been true in the past, or will be true in the future. We could go on and on, but let's stop now for some exercises.

9. Exercises: Construct frameworks of world-models demonstrating the invalidity of the following sequents:

Part I:

(1) Show that the D-schema ($\Box\varphi \models \Diamond\varphi$) does not hold in K.
(2) Show that the T-schema ($\Box\varphi \models \varphi$) does not hold in D.
(3) Show that the B-schema ($\Diamond\Box\varphi \models \varphi$) does not hold in T.
(4) Show that the S4-schema ($\Box\varphi \models \Box\Box\varphi$) does not hold in T.
(5) Show that the B-schema ($\Diamond\Box\varphi \models \varphi$) does not hold in S4.
(6) Show that the S4-schema ($\Box\varphi \models \Box\Box\varphi$) does not hold in B.
(7) Show that the S5-schema ($\Diamond\varphi \models \Box\Diamond\varphi$) holds in neither S4 nor B.

Part II:

From what you have shown in part 1, diagram the relative strengths of these systems, starting with K as the weakest, and working your way up to S5 as the strongest.

Part III:

Identify some sequent (other than $\Diamond\varphi \models \Box\Diamond\varphi$) that would be valid in S5, but not S4, and then show why that would be.